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Resonantly interacting solitary waves

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Resonant (phase-locked) interactions among three obliquely oriented solitary waves are studied. It is shown that such interactions are associated with the parametric end points of the singular regime for interactions between two solitary waves. The latter include regular reflexion at a rigid wall, which is impossible for $\psi_i < (3\alpha)^{\frac{1}{2}}$ (ψ_i = angle of incidence, α = amplitude/depth $\ll 1$), and it is shown that the observed phenomenon of 'Mach reflexion' can be described as a resonant interaction in this regime. The run-up at the wall is calculated as a function of $\psi_i/(3\alpha)^{\frac{1}{2}}$ and is found to have a maximum value of $4\alpha d$ for $\psi_i = (3\alpha)^{\frac{1}{2}}$. This same resonant interaction also describes diffraction of a solitary wave at a corner of internal angle $\pi - \psi_i$, $-(3\alpha)^{\frac{1}{2}} < \psi_i < (3\alpha)^{\frac{1}{2}}$, and suggests that a solitary wave cannot turn through an angle in excess of $(3\alpha)^{\frac{1}{2}}$ at a convex corner without separating or otherwise losing its identity.

1. Introduction

A solitary wave (*soliton*) of free-surface displacement $\alpha d\eta$ in water of quiescent depth d has the dimensionless description

$$\eta = k^2 \operatorname{sech}^2 \theta + O(\alpha), \tag{1.1}$$

where

$$\theta = \mathbf{k}.\mathbf{x} - \omega t + \theta_0, \tag{1.2}$$

$$\mathbf{k} = k\{\cos\psi, \sin\psi\}, \quad \omega \equiv kc = k\{1 + \tfrac{1}{2}k^2\alpha + O(\alpha^2)\} \qquad (1.3a, b)$$

are the phase, wavenumber and circular frequency, c is the wave speed, θ_0 is a phase constant, $\mathbf{x} \equiv \{x, z\}$ is the co-ordinate vector in a horizontal plane,

$$l = 2(3\alpha)^{-\frac{1}{2}} d \equiv \beta^{-\frac{1}{2}} d, \tag{1.4}$$

$l/(gd)^{\frac{1}{2}}$ and $(gd)^{\frac{1}{2}}$ are the reference values of length, time and speed, and α and $\beta \equiv \frac{3}{4}\alpha$ are small parameters. The subscript n is appended to η, k, ψ, c, ω, θ and θ_0 in the following treatment of interacting solitons, and k_n^2 then appears as the relative amplitude (we may choose $k \equiv 1$ for a single soliton, or for one member of a set of solitons, by choosing α such that αd is the maximum displacement for that soliton).

The oblique interaction between two solitons is described by (Miles 1977, hereinafter referenced by I, followed by the appropriate equation or section number)

$$\tfrac{1}{4}\eta = \frac{k_1^2 E_1 + k_2^2 E_2 + (k_1 - k_2)^2 E_1 E_2 + e^{2\delta}\{(k_1+k_2)^2 + k_2^2 E_1 + k_1^2 E_2\} E_1 E_2}{(1 + E_1 + E_2 + e^{2\delta} E_1 E_2)^2}, \tag{1.5}$$

where
$$E_n = \exp(-2\theta_n), \tag{1.6}$$
$$\delta = \tfrac{1}{2}\log\left\{\frac{\sin^2\psi - \beta(k_1-k_2)^2}{\sin^2\psi - \beta(k_1+k_2)^2}\right\}, \tag{1.7}$$
$$\psi = \tfrac{1}{2}(\psi_2 - \psi_1) \qquad (1.8)\dagger$$

and, here and subsequently, error factors of $1+O(\alpha)$ are implicit. The parameters $\beta(k_1+k_2)^2$, $\beta(k_1-k_2)^2$ and ψ are, respectively, measures of mean strength, relative strength, and obliquity. The individual solutions η_1 and η_2 may be superimposed if $\psi^2 \gg 3\alpha$, for then $\delta = 1+O(\alpha)$, and (1.5) reduces to $\eta = \eta_1+\eta_2$ to within $1+O(\alpha)$. Superposition fails if $\psi^2 = O(\alpha)$, and the interaction between two incoming solitons η_1 and η_2 then yields outgoing solitons with phase shifts of magnitude δ and signs depending on the relative values of ψ^2 and $(k_2^2-k_1^2)\alpha$; see I § 6 for details.

Perhaps the most striking feature of the interaction described by (1.5) is that it is singular if
$$\beta(k_1-k_2)^2 < \psi^2 < \beta(k_1+k_2)^2 \tag{1.9}$$
in the general (asymmetric) case or if
$$0 < \psi^2 < 3\alpha \tag{1.10}$$
for reflexion at a rigid wall, for which $k_1 = k_2 \equiv 1$ and $\psi_2 = -\psi_1 \equiv \psi$.

I now proceed to show that the end points of this singular regime, $\psi^2 = \beta(k_1 \mp k_2)^2$, are associated with resonant interactions among three solitons.‡ I then go on (in § 4) to show that such an interaction provides an asymptotic (in time or downstream distance) solution of the problem of 'Mach reflexion' (Wiegel 1964*a*, *b*) of a solitary wave if $\psi_i^2 < 3\alpha$, where ψ_i is the angle of incidence. The ratio of the maximum free-surface displacement at the wall to the amplitude of the incident wave is a relatively simple function of $\psi_i/(3\alpha)^{\frac{1}{2}}$ that increases from the value of 2 predicted by linearized theory to a maximum of 4 at $\psi_i^2 = 3\alpha$ and then decreases to 1 as $\psi_i^2/3\alpha \downarrow 0$ (corresponding to a wave moving parallel to the wall). This result may be of some practical significance in connexion with tsunamis (Wiegel 1964*b*).

The solution developed in § 4 also provides an asymptotic description of the diffraction of a soliton at a corner of internal angle $\pi - \psi_i$, $-(3\alpha)^{\frac{1}{2}} < \psi_i < (3\alpha)^{\frac{1}{2}}$, and suggests that a soliton cannot turn through an angle in excess of $(3\alpha)^{\frac{1}{2}}$ at a convex corner without separating or otherwise losing its identity.

It must be emphasized that the present theory is based on the limit $\alpha \downarrow 0$ (weak nonlinearity). The available experimental data (Perroud 1957; Chen 1961) are for non-small α (0·2–0·6) and, in this and other respects, are inadequate for a quantitative test of the present theory. The observed patterns are in qualitative agreement with those predicted here; however, the data suggest that the critical value of ψ_i [$(3\alpha)^{\frac{1}{2}}$ according to the present theory] may tend to a constant

† The unsubscripted parameter ψ is defined by (1.8), and ψ as defined by (1.3*a*) always appears with a subscript, throughout the sequel.

‡ Resonant interactions among three unidirectional solitons are considered by Kaup (1976).

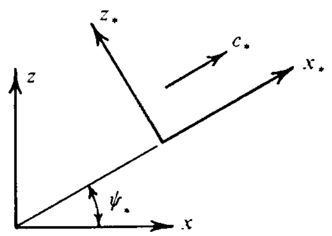

FIGURE 1. The moving reference frame R_*.

value of roughly 45° with increasing α, although breaking naturally occurs for sufficiently large α.

2. Wave kinematics

We define a resonant (*phase-locked* might be more precise) interaction among three solitons, the phases of which are defined by (1.2) with subscripts appended, by the conditions

$$\mathbf{k}_3 = \mathbf{k}_2 \pm \mathbf{k}_1, \quad \omega_3 = \omega_2 \pm \omega_1, \tag{2.1 a, b}$$

where, here and subsequently, the signs are vertically ordered (note that reversing this order is equivalent to interchanging the subscripts 2 and 3), and the subscripts $\pm$ in the sequel refer to the corresponding alternatives. Substituting (1.3) into (2.1) and letting $\alpha \downarrow 0$ with $\mathbf{k}_{1,2}$ prescribed yields the resonance conditions

$$k_3 = k_2 \pm k_1, \quad k_3 \psi_3 = k_2 \psi_2 \pm k_1 \psi_1 \tag{2.2 a, b}$$

and
$$\psi^2 \equiv \tfrac{1}{4}(\psi_2 - \psi_1)^2 = \beta(k_2 \pm k_1)^2 \equiv \psi_\pm^2. \tag{2.2 c}$$

We emphasize that (2.1) can be satisfied only if $\psi^2 = O(\alpha)$ and that, as anticipated in § 1, (2.2c) corresponds to the end points in (1.9).

The phases θ_1 and θ_2, and hence the solitons η_1 and η_2, are stationary in a reference frame R_* moving with a velocity $\mathbf{c}_* \equiv c_*\{\cos\psi_*, \sin\psi_*\}$ that is determined by

$$c_n \sec(\psi_n - \psi_*) = c_* \tag{2.3}$$

for $n = 1$ and 2 (the projection of $\mathbf{c}_*$ on $\mathbf{k}_n$, i.e. on the normal to the surface $\theta_n =$ constant, must be equal to c_n; cf. Snell's law). It then follows from (2.1) that (2.3) holds also for $n = 3$, by virtue of which the resonant interaction is stationary in R_*. Introducing (see figure 1)

$$x_* = x\cos\psi_* + z\sin\psi_* - c_* t, \quad z_* = -x\sin\psi_* + z\cos\psi_* \tag{2.4 a, b}$$

in (1.2) and invoking $\psi_n - \psi_* = O(\alpha^{\frac{1}{2}})$ yields

$$\theta_n - \theta_{0n} = k_n\{x_* + (\psi_n - \psi_*) z_*\} \tag{2.5}$$

for the phases in R_* within the present approximation.

Substituting c_n from (1.3) into (2.3) and solving (with $n = 1, 2$) for c_* and ψ_* yields

$$c_* = 1 + \tfrac{2}{3}\alpha(k_1^2 + k_2^2 \pm k_1 k_2) \tag{2.6 a}$$

and
$$\psi_* = \tfrac{1}{2}(\psi_2 + \psi_1) + \tfrac{1}{2}\alpha(k_2^2 - k_1^2)(\psi_2 - \psi_1)^{-1}. \tag{2.6 b}$$

$\theta_1 \sim$	$\theta_2 \sim$	$\theta_3 \sim$	$z_* \sim$		$x_* \sim$				$\eta \sim$
					$k_2 < \frac{1}{2}k_1$	$\frac{1}{2}k_1 < k_2 < k_1$	$k_1 < k_2 < 2k_1$	$k_2 > 2k_1$	
$O(1)$	$\pm\infty$	$\pm\infty$	$\pm\infty$		$\pm\infty$	$\mp\infty$	$\pm\infty$	$\pm\infty$	η_1 $o(1)$
$\pm\infty$	$O(1)$	$\mp\infty$	$\mp\infty$		$\pm\infty$	$\pm\infty$	$\mp\infty$	$\pm\infty$	η_2 $o(1)$
			$k_2 < k_1$	$k_2 > k_1$					
$\mp\infty$	$\mp\infty$	$O(1)$	$\mp\infty$	$\pm\infty$	$\mp\infty$				η_3 $o(1)$

TABLE 1. The asymptotic limits associated with (3.5) and figure 2.

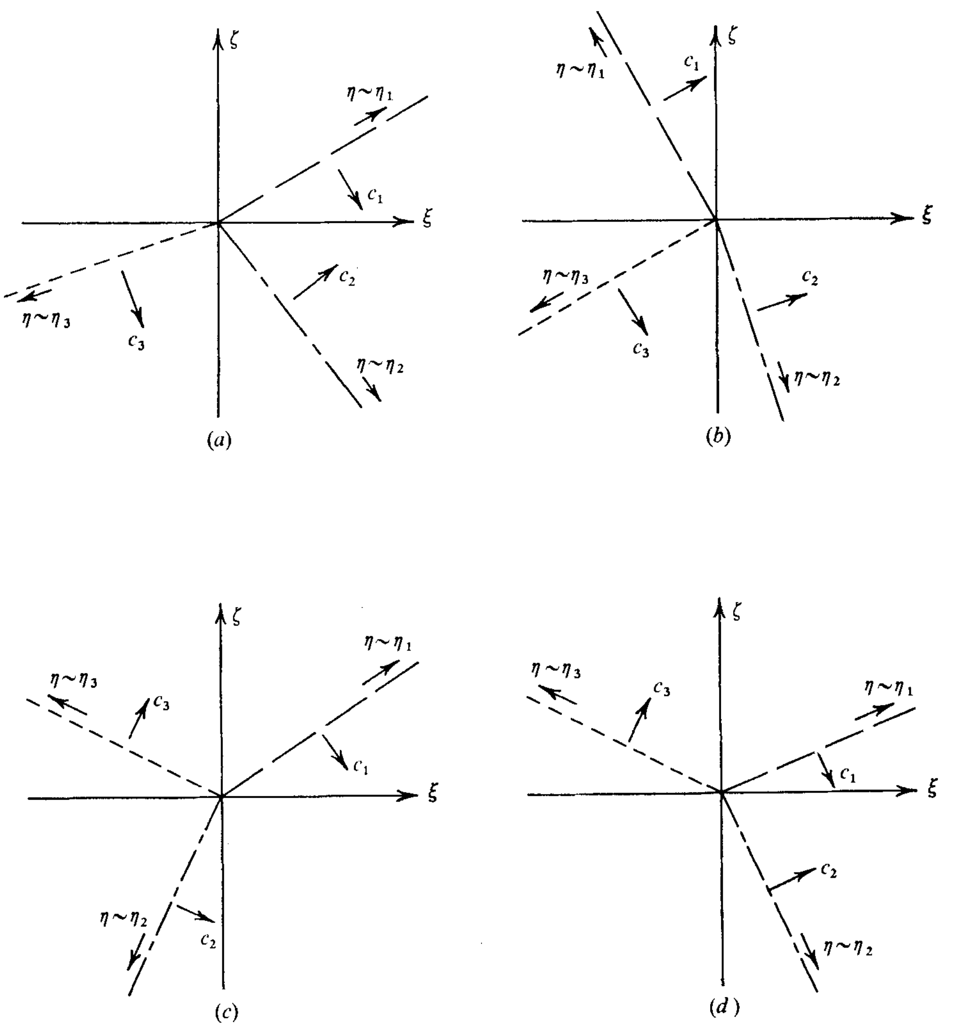

FIGURE 2. The resonant interactions described by (3.3)–(3.5) in R_* for (a) $k_2 < \frac{1}{2}k_1$, (b) $\frac{1}{2}k_1 < k_2 < k_1$, (c) $k_1 < k_2 < 2k_1$ and (d) $k_2 > 2k_1$. The broken lines are surfaces of constant θ_1 (— — — —), θ_2 (— · —) and θ_3 (- - -). The angular scale is exaggerated by the transformation (3.1).

Invoking (2.2) yields the more symmetrical expressions

$$c_* = 1+\tfrac{1}{3}\alpha(k_1^2+k_2^2+k_3^2), \quad \psi_* = \tfrac{1}{3}(\psi_1+\psi_2+\psi_3). \tag{2.7a,b}$$

3. Solutions

We now align the x axis with $\mathbf{c}_*$ and introduce the rescaled co-ordinates

$$\xi = x - c_* t, \quad \zeta = (3\alpha)^{\frac{1}{2}} z \quad (\psi_* \equiv 0) \tag{3.1a,b}$$

(the scale of the interaction zone is $l \sim d/\alpha^{\frac{1}{2}}$ in the direction of $\mathbf{c}_*$ and $l/(3\alpha)^{\frac{1}{2}} \sim d/\alpha$ in the transverse direction). We do not require the equations of motion (see I § 2) explicitly for the present development, but it is worth noting that introducing (3.1) in I (2.5) and I (2.6), letting $\alpha \downarrow 0$ with $\xi, \zeta = O(1)$, and assuming that η is stationary in R_* yields

$$\eta_{\xi\xi\xi\xi} + 6(\eta^2)_{\xi\xi} - 8k_*^2\,\eta_{\xi\xi} + 12\eta_{\zeta\zeta} = 0, \tag{3.2}$$

where $k_*^2 = \frac{1}{3}(k_1^2+k_2^2+k_3^2)$.

We obtain the solution of (3.2) for $\psi = \psi_-$ by letting $\delta \downarrow -\infty$ in (1.5):

$$\tfrac{1}{4}\eta = \frac{k_1^2 \exp(-2\theta_1) + k_2^2 \exp(-2\theta_2) + (k_1-k_2)^2 \exp\{-2(\theta_1+\theta_2)\}}{[1+\exp(-2\theta_1)+\exp(-2\theta_2)]^2} \quad (\psi = \psi_-). \tag{3.3}$$

To show that (3.3) corresponds to the resonant interaction defined by $(2.2)_-$, we assume (for definiteness) that $\psi_2 > \psi_1$, solve $(2.2c)_-$, (2.6b) and (2.7b) for ($\psi_n - \psi_*$ in § 2 $\equiv \psi_n$ here)

$$\{\psi_1, \psi_2, \psi_3\} = (\tfrac{1}{3}\alpha)^{\frac{1}{2}}\{k_1 - 2k_2, k_2 - 2k_1, k_1+k_2\}\,\mathrm{sgn}\,(k_2-k_1) \quad (\psi = \psi_- > 0), \tag{3.4}$$

combine the result with (2.5) in (3.3), and carry out the limits (with $k_{1,2} > 0$) summarized in table 1 (cf. I § 6). It follows from these limits that (3.3) describes the resonant interactions (see figure 2)

$$\{\eta_1, \eta_2\} \rightarrow \eta_3 \quad (k_2 < \tfrac{1}{2}k_1), \tag{3.5a}$$

$$\eta_2 \rightarrow \{\eta_1, \eta_3\} \quad (\tfrac{1}{2}k_1 < k_2 < k_1), \tag{3.5b}$$

$$\eta_1 \rightarrow \{\eta_2, \eta_3\} \quad (k_1 < k_2 < 2k_1) \tag{3.5c}$$

and

$$\{\eta_1, \eta_2\} \rightarrow \eta_3 \quad (k_2 > 2k_1), \tag{3.5d}$$

where the left/right-hand sides correspond to the incoming/outgoing waves at large distances from the interaction zone.

The marginal case $k_2 = \frac{1}{2}k_1$ yields $\psi_1 = 0$ and $c_1 = c_*$, such that an observer in R_* perceives $\eta \sim \eta_1/o(1)$ on his left/right and $\eta \sim \eta_2/\eta_3$ in his fourth/third quadrants. The marginal case $k_2 = 2k_1$ yields $\psi_2 = 0$ and $c_2 = c_*$, such that an observer in R_* perceives $\eta \sim o(1)/\eta_2$ on his left/right and $\eta \sim \eta_1/\eta_3$ in his first/second quadrants. (The marginal case $k_1 = k_2$ corresponds to a single wave and is trivial in the present context.)

$\theta_1 \sim$	$\theta_2 \sim$	$\theta_3 \sim$	$z_* \sim$	$x_* \sim$		$\eta \sim$
$O(1)$	$\pm\infty$	$\pm\infty$	$\pm\infty$	$\pm\infty$		η_1 $o(1)$
$\mp\infty$	$O(1)$	$\mp\infty$	$\pm\infty$	$\mp\infty$		η_2 $o(1)$
				$k_2 < k_1$	$k_2 > k_1$	
$\pm\infty$	$\mp\infty$	$O(1)$	$\mp\infty$	$\mp\infty$	$\pm\infty$	η_3 $o(1)$

TABLE 2. The asymptotic limits associated with (3.8) and figure 3

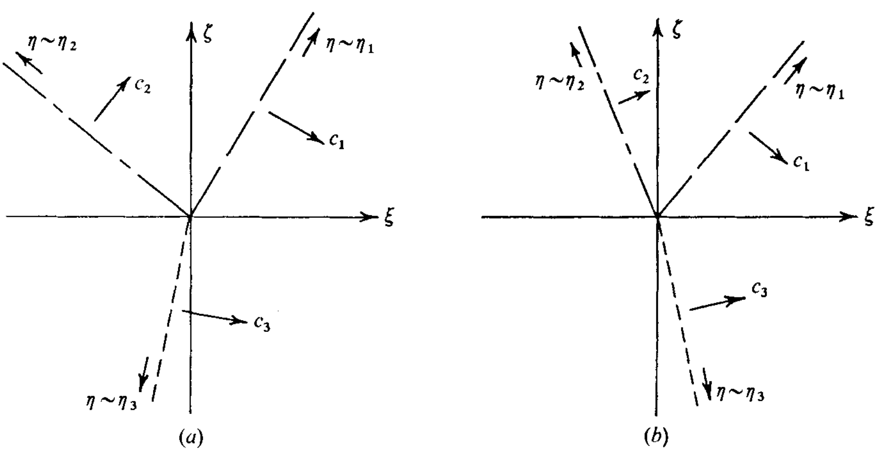

FIGURE 3. The resonant interactions described by (3.6)–(3.8) for (a) $k_2 < k_1$ and (b) $k_2 > k_1$. The broken lines are surfaces of constant θ_1 (— — — —), θ_2 (— - —) and θ_3 (- - - -). The angular scale is exaggerated by the transformation (3.1).

We obtain the solution of (3.2) for $\psi = \psi_+$ simply by changing the signs of both k_1 and θ_1 in (3.3) or, equivalently, interchanging the subscripts 2 and 3:

$$\tfrac{1}{4}\eta = \frac{k_1^2 \exp(2\theta_1) + k_2^2 \exp(-2\theta_2) + (k_1+k_2)^2 \exp\{2(\theta_1-\theta_2)\}}{[1+\exp(2\theta_1)+\exp(-2\theta_2)]^2} \tag{3.6a}$$

$$= \frac{k_1^2 \exp(-2\theta_1) + k_3^2 \exp(-2\theta_3) + (k_3-k_1)^2 \exp\{-2(\theta_1+\theta_3)\}}{[1+\exp(-2\theta_1)+\exp(-2\theta_3)]^2} \quad (\psi = \psi_+), \tag{3.6b}$$

where θ_n is given by (2.5) with

$$\{\psi_1, \psi_2, \psi_3\} = (\tfrac{1}{3}\alpha)^{\frac{1}{2}}\{-(k_1+2k_2), k_2+2k_1, k_2-k_1)\} \quad (\psi = \psi_+ > 0). \tag{3.7}$$

Carrying out the limits summarized in table 2 yields (see figure 3)

$$\eta_1 \to \{\eta_2, \eta_3\} \quad (k_2 < k_1), \tag{3.8a}$$

$$\{\eta_1, \eta_3\} \to \eta_2 \quad (k_2 > k_1). \tag{3.8b}$$

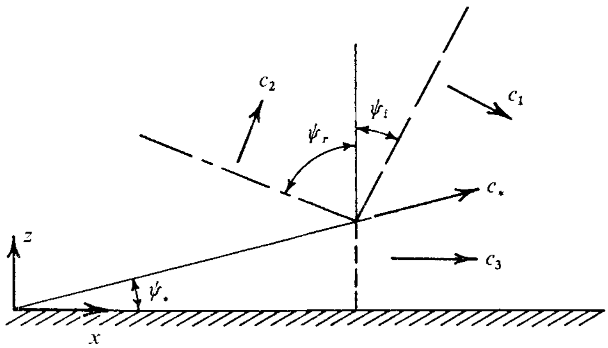

FIGURE 4. The Mach-reflexion pattern of § 4. The angular scale is exaggerated.

4. Mach reflexion

Observation (Wiegel 1964*a*, *b*) and experiment (Perroud 1957; Chen 1961) reveal that regular reflexion of a solitary wave at a rigid wall (for which $k_2 = k_1$, $\psi_2 = -\psi_1 \equiv \psi$ and $\psi_* = 0$ in the present notation) is impossible for sufficiently small angles of incidence and is replaced by 'Mach reflexion' (geometrically similar to the corresponding shock-wave reflexion). The apex of the incident and reflected waves then moves away from the wall at a constant angle, say ψ_*, and is joined to the wall by a third solitary wave (the 'Mach stem'), as shown in figure 4. Moreover, the strength of the reflected wave decreases to zero with the angle of incidence. There is some question as to the stability of the resulting waves, and the observed stem-wave profile may depart significantly from that of a true (Boussinesq) solitary wave, but the available data are not definitive. It seems likely, nevertheless, that there exists a parametric regime in which the Mach-reflexion pattern is realized and that the pattern is asymptotically stationary (the reflexion is initiated at the leading edge of a wall of finite length in the experiments, and non-stationary effects must be significant near the leading edge).

Against this background, we consider the resonant interaction described by (3.6), (3.7) and (3.8*a*) with η_1 as the incident wave, η_2 as the reflected wave and η_3 as the stem wave. Replacing ψ_n in § 3 by $\psi_n - \psi_*$ (as in § 2 and such that ψ_n is now measured from the wall), choosing $k_1 \equiv 1$, $\psi_1 \equiv -\psi_i$ and $\psi_3 \equiv 0$, and invoking $(2.2)_+$ and (2.7) yields

$$\{k_1, k_2, k_3\} = \{1, k, 1+k\}, \quad \{\psi_1, \psi_2, \psi_3\} = (3\alpha)^{\frac{1}{2}}\{-k, 1, 0\}, \qquad (4.1\,a, b)$$

$$\psi_* = (\tfrac{1}{3}\alpha)^{\frac{1}{2}}(1-k), \quad c_* = 1 + \tfrac{2}{3}\alpha(1+k+k^2) \qquad (4.2\,a, b, c)$$

and

$$k = \psi_i/(3\alpha)^{\frac{1}{2}}. \qquad (4.3)$$

Choosing $\theta_{n0} \equiv 0$, such that $\theta_n = 0$ at $x = z = t = 0$ for $n = 1$, 2 and 3, and introducing the R_* co-ordinates of (2.4) yields

$$\{\theta_1, \theta_2, \theta_3\} = \{1, k, 1+k\}x_* + \{-(1+2k), k(2+k), -(1-k^2)\}(\tfrac{1}{3}\alpha)^{\frac{1}{2}} z_*. \qquad (4.4)$$

Requiring the apex to move away from the wall ($\psi_* > 0$) implies $k < 1$, so that

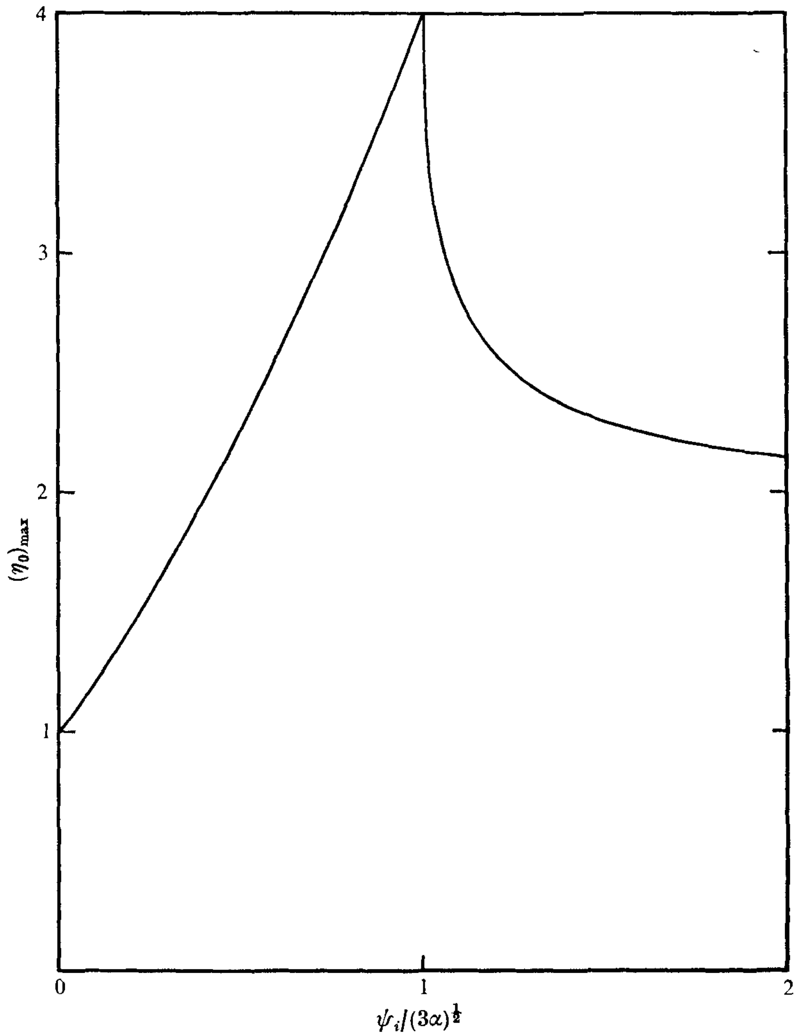

FIGURE 5. The run-up at the wall, as given by (4.5).

the parametric range of interest is $0 < k < 1$ in the present context (but see last paragraph in this section). The limit $k \uparrow 1$ ($\psi_* \downarrow 0$) corresponds to regular reflexion for $\psi^2 = 3\alpha$. The limit $k \downarrow 0$ ($\psi_i \uparrow 0$, $k_2 \downarrow 0$) corresponds to a single wave, $\eta = \operatorname{sech}^2(\theta_1 - \frac{1}{2}\ln 2)$, moving parallel to the wall.

Substituting (4.4) into (3.6) yields the asymptotic solution as $c_* t \to \infty$. We emphasize that this solution satisfies the boundary condition of zero transverse velocity at the wall only asymptotically, whereas the solution given by (1.5) for $k_1 = k_2 = 1$ and $\psi_2 = -\psi_1 = \psi > (3\alpha)^{\frac{1}{2}}$ satisfies it exactly.

It follows from (4.1) and (4.3) that the dimensionless amplitude $k^2 = \psi_i^2/3\alpha$ of the reflected wave decreases from 1 to 0 as ψ_i decreases from $(3\alpha)^{\frac{1}{2}}$ to 0, in qualitative agreement with observation, whilst the angle of reflexion remains at $(3\alpha)^{\frac{1}{2}}$. The amplitude $(1+k)^2$ of the stem wave, and therefore the asymptotic amplitude at the wall, decreases from 4 to 1 in the same interval. Combining this result with that for regular reflexion, I (5.12), yields

$$(\eta_0)_{\max} = \begin{cases} 4[1+\{1-(3\alpha/\psi_i^2)\}^{\frac{1}{2}}]^{-1} & (\psi_i^2 > 3\alpha) \quad (4.5a) \\ \{1+(3\alpha)^{-\frac{1}{2}}\psi_i\}^2 & (\psi_i^2 < 3\alpha) \quad (4.5b) \end{cases}$$

for the amplitude at the wall (see figure 5). It follows that the maximum run-up of a weakly nonlinear solitary wave incident upon a rigid wall occurs for $\psi_i = (3\alpha)^{\frac{1}{2}}$ and is twice that predicted by linearized theory [$(\eta_0)_{\max} \downarrow 2$ for $\alpha \downarrow 0$ in (4.5a)].

The preceding solution, developed in the context of reflexion, also provides the asymptotic solution for the diffraction of a solitary wave at a concave corner of internal angle $\pi - \psi_i$, $0 \leqslant \psi_i \leqslant (3\alpha)^{\frac{1}{2}}$, or, equivalently, by a wedge of angle $2\psi_i$ [the solution for regular reflexion provides the corresponding diffraction solution if $\psi_i > (3\alpha)^{\frac{1}{2}}$].

The solution for $-1 < k < 0$ provides a solution for diffraction at a convex corner of internal angle $\pi - k(3\alpha)^{\frac{1}{2}}$. The limit $k \downarrow -1$ corresponds to a single wave ($\psi_2 = \psi_1$) that vanishes asymptotically ($k_3 = 0$) below $dy/dx = 2(\frac{1}{3}\alpha)^{\frac{1}{2}}$. This suggests that a soliton cannot turn through an angle greater than $(3\alpha)^{\frac{1}{2}}$ at a convex corner without separating or otherwise losing its identity.

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